Homotopy Type Theory

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- New constructions based on homotopical intuitions are added as Higher Inductive Types, providing many classical spaces, quotient types, truncations, etc.
- The new Univalence Axiom is also added. It implies that isomorphic structures are equal, in a certain sense.

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- Applications to software verification are current work in progress.
- There is a comprehensive book containing the informal development, which was written at a year-long special research program at the Institute for Advanced Study in Princeton.

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Formal calculus of typed terms and equations, presented as a deductive system by rules of inference.
Intended as a foundation for constructive mathematics, but now used also in the theory of programming languages and as the basis of computational proof assistants.

Propositions as Types

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This is known as the **Curry-Howard correspondence**:

0	1	$A + B$	$A \times B$	$A \rightarrow B$	$\sum_{x:A} B(x)$	$\prod_{x:A} B(x)$
$\bot$	$\top$	$A \vee B$	$A \wedge B$	$A \Rightarrow B$	$\exists_{x:A} B(x)$	$\forall_{x:A} B(x)$

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Gives the system its **constructive character**.

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$$\frac{c : \mathtt{Id}_A(a, b) \qquad x : A \ \vdash \ d(x) : R\big(x, x, \mathtt{r}(x)\big)}{\mathtt{J}_d(a, b, c) : R(a, b, c)}$$

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Schematically:

$$\text{“ } a = b \ \& \ R(x, x) \ \Rightarrow \ R(a, b) \text{ ”}$$

Intensionality

The rules are such that if a and b are **equal** as terms:

$$a = b$$

then they are also logically **identical**:

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- Allowing such distinctions gives the system good computational and proof-theoretic properties.
- It also gives rise to a structure of great combinatorial complexity.

The homotopy interpretation (Awodey-Warren)

Suppose we have terms of ascending identity types:

$$\begin{aligned} a,\ b &: A \\ p,\ q &: \mathtt{Id}_A(a, b) \\ \alpha,\ \beta &: \mathtt{Id}_{\mathtt{Id}_A(a,b)}(p, q) \\ \ldots &: \mathtt{Id}_{\mathtt{Id}_{\mathtt{Id}_{\ldots}}}(\ldots) \end{aligned}$$

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Consider the following interpretation:

$$
\begin{array}{rcl}
\text{Types} & \rightsquigarrow & \text{Spaces} \\
\text{Terms} & \rightsquigarrow & \text{Maps} \\
a : A & \rightsquigarrow & \text{Points } a : 1 \to A \\
p : \mathtt{Id}_A(a,b) & \rightsquigarrow & \text{Paths } p : a \Rightarrow b \\
\alpha : \mathtt{Id}_{\mathtt{Id}_A(a,b)}(p,q) & \rightsquigarrow & \text{Homotopies } \alpha : p \Rrightarrow q \\
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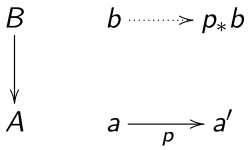

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$$\begin{array}{ccc} B & \quad & b \cdots\cdots\!\!\!\!> p_* b \\ \Big\downarrow & & \\ A & & a \xrightarrow[p]{\qquad} a' \end{array}$$

This is the notion of a "fibration" of spaces.

The homotopy interpretation: Type dependency

Thus we continue the homotopy interpretation as follows:

$$\text{Dependent types} \quad x : A \vdash B(x) \quad \rightsquigarrow \quad \text{Fibrations} \quad \begin{array}{c} B \\ \downarrow \\ A \end{array}$$

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The type $B(a)$ is the fiber of $B \to A$ over the point $a : A$

$$\begin{array}{ccc} B(a) & \longrightarrow & B \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[a]{} & A. \end{array}$$

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The fiber $\mathtt{Id}_A(a, b)$ over a point $(a, b) \in A \times A$ is the space of paths from a to b in A.

$$\begin{array}{ccc} \mathtt{Id}_A(a, b) & \longrightarrow & A^I \\ \downarrow & & \downarrow \\ 1 & \xrightarrow[(a,b)]{} & A \times A. \end{array}$$

The homotopy interpretation: Identity types

The path space A^I classifies homotopies $\vartheta : f \Rightarrow g$ between maps $f, g : X \to A$,

$$\begin{array}{ccc} & & A^I \\ & \overset{\vartheta}{\nearrow} & \downarrow \\ X & \xrightarrow[(f,g)]{} & A \times A. \end{array}$$

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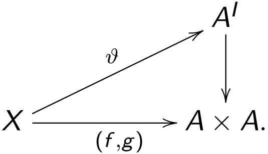

So given any terms $x : X \vdash f, g : A$, an identity term

$$x : X \vdash \vartheta : \mathtt{Id}_A(f, g)$$

is interpreted as a **homotopy** between f and g.

The homotopy interpretation: Summary

This takes the familiar **topological interpretation** of the *simply-typed* λ-calculus:

$$\text{types} \rightsquigarrow \text{spaces}$$
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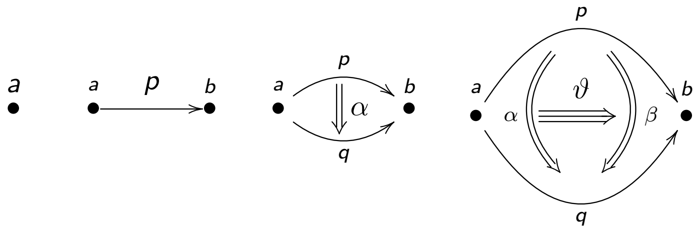
This forces:

- *dependent types to be fibrations*,
- `Id`*-types to be path spaces*,
- *homotopic maps to be identical*.

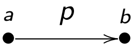
The fundamental groupoid of a type (Hofmann-Streicher)

Like path spaces in topology, identity types endow each type in the system with the structure of a (higher-) groupoid:



Fundamental groupoids

As in topology, the terms of order 0 and 1, ("points" and "paths") bear the structure of a **groupoid**.



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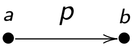
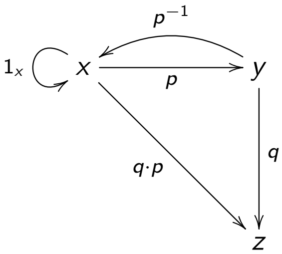


Definition

A *groupoid* is a category in which every arrow has an inverse.



The fundamental groupoid of a type

The laws of identity are then the **groupoid operations**:

$r : \mathtt{Id}(a, a)$	reflexivity	$a \rightarrow a$
$s : \mathtt{Id}(a, b) \rightarrow \mathtt{Id}(b, a)$	symmetry	$a \leftrightarrows b$
$t : \mathtt{Id}(a, b) \times \mathtt{Id}(b, c) \rightarrow \mathtt{Id}(a, c)$	transitivity	$a \rightarrow b \rightarrow c$

The fundamental groupoid of a type

But also just as in topology, the **groupoid equations** of associativity, inverse, and unit:

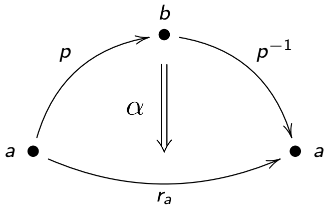
$$p \cdot (q \cdot r) = (p \cdot q) \cdot r$$
$$p^{-1} \cdot p = 1 = p \cdot p^{-1}$$
$$1 \cdot p = p = p \cdot 1$$

do not hold strictly, but only "up to homotopy".

The fundamental groupoid of a type

This means they are witnessed by terms of the next higher order:

$$\alpha : \mathtt{Id}_{\mathtt{Id}}\left(p^{-1} \cdot p, r_a\right)$$



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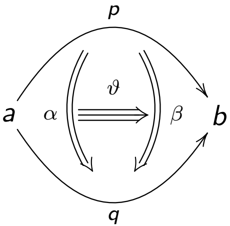
In this way, each type in the system is endowed with the structure of an "∞-groupoid", with terms, identities between terms, identities between identities, ...



Homotopy n-types (Voevodsky)

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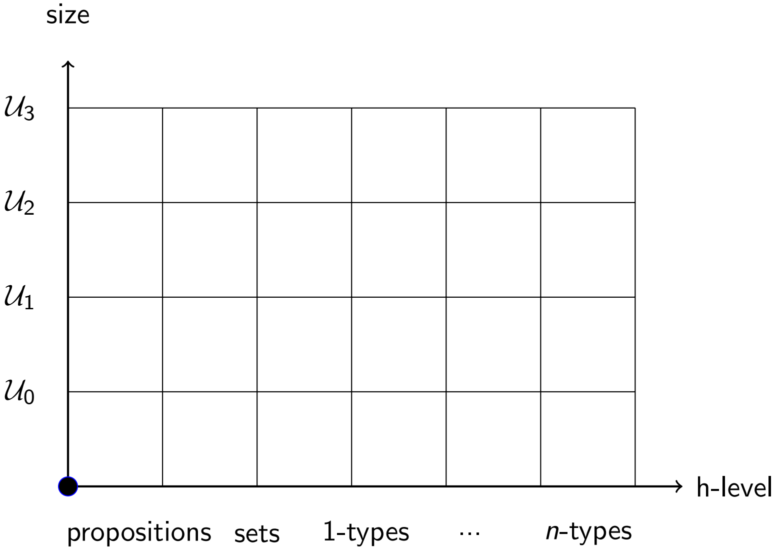
(n+1)-type iff $\prod_{x,y:X} \mathsf{nType}(\mathtt{Id}_X(x,y))$.

The Hierarchy of n-Types

This gives a new view of the mathematical universe, in which some types have intrinsic higher-dimensional structure.



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This allows for computer verified proofs in homotopy theory and related fields, in addition to classical and constructive mathematics. This is being very actively pursued right now.

A computational example

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This argument can be formalized in Coq and checked by a computer. In this way, we can use the homotopical interpretation to give machine-checked proofs in homotopy theory.

A computational example

```
(** ** The 2-dimensional groupoid structure *)

(** Horizontal composition of 2-dimensional paths. *)
Definition concat2 {A} {x y z : A} {p p' : x = y} {q q' : y = z} (h : p = p') (h' : q = q')
: p @ q = p' @ q'
:= match h, h' with idpath, idpath => 1 end.

Notation ``p @@ q" := (concat2 p q)

(** 2-dimensional path inversion *)
Definition inverse2 {A : Type} {x y : A} {p q : x = y} (h : p = q) : p^ = q^
:= match h with idpath => 1 end.

(** *** Whiskering *)

Definition whiskerL {A : Type} {x y z : A} (p : x = y) {q r : y = z} (h : q = r) : p @ q = p @ r
:= 1 @@ h.

Definition whiskerR {A : Type} {x y z : A} {p q : x = y} (h : p = q) (r : y = z) : p @ r = q @ r
:= h @@ 1.

(** *** Unwhiskering, a.k.a. cancelling. *)

Lemma cancelL {A} {x y z : A} (p : x = y) (q r : y = z) : (p @ q = p @ r) -> (q = r).
Proof.
  destruct p, r. intro a. exact ((concat_1p q)^ @ a).
Defined.

Lemma cancelR {A} {x y z : A} (p q : x = y) (r : y = z) : (p @ r = q @ r) -> (p = q).
Proof.
  destruct r, p. intro a. exact (a @ concat_p1 q).
Defined.
```

```
(** Whiskering and identity paths. *)

Definition whiskerR_p1 {A : Type} {x y : A} {p q : x = y} (h : p = q) :
  (concat_p1 p) ^ @ whiskerR h 1 @ concat_p1 q = h
  :=
  match h with idpath =>
    match p with idpath =>
      1
    end end.

Definition whiskerR_1p {A : Type} {x y z : A} (p : x = y) (q : y = z) :
  whiskerR 1 q = 1 :> (p @ q = p @ q)
  :=
  match q with idpath => 1 end.

Definition whiskerL_p1 {A : Type} {x y z : A} (p : x = y) (q : y = z) :
  whiskerL p 1 = 1 :> (p @ q = p @ q)
  :=
  match q with idpath => 1 end.

Definition whiskerL_1p {A : Type} {x y : A} {p q : x = y} (h : p = q) :
  (concat_1p p) ^ @ whiskerL 1 h @ concat_1p q = h
  :=
  match h with idpath =>
    match p with idpath =>
      1
    end end.

Definition concat2_p1 {A : Type} {x y : A} {p q : x = y} (h : p = q) :
  h @@ 1 = whiskerR h 1 :> (p @ 1 = q @ 1)
  :=
  match h with idpath => 1 end.

Definition concat2_1p {A : Type} {x y : A} {p q : x = y} (h : p = q) :
  1 @@ h = whiskerL 1 h :> (1 @ p = 1 @ q)
  :=
  match h with idpath => 1 end.
```

```
(** The interchange law for concatenation. *)
Definition concat_concat2 {A : Type} {x y z : A} {p p' p'' : x = y} {q q' q'' : y = z}
  (a : p = p') (b : p' = p'') (c : q = q') (d : q' = q'') :
  (a @@ c) @ (b @@ d) = (a @ b) @@ (c @ d).
Proof.
  case d.
  case c.
  case b.
  case a.
  reflexivity.
Defined.

(** The interchange law for whiskering.  Special case of [concat_concat2]. *)
Definition concat_whisker {A} {x y z : A} (p p' : x = y) (q q' : y = z) (a : p = p') (b : q = q') :
  (whiskerR a q) @ (whiskerL p' b) = (whiskerL p b) @ (whiskerR a q')
  :=
  match b with
    idpath =>
    match a with idpath =>
      (concat_1p _)^
    end
  end.

(** Structure corresponding to the coherence equations of a bicategory. *)

(** The "pentagonator": the 3-cell witnessing the associativity pentagon. *)
Definition pentagon {A : Type} {v w x y z : A} (p : v = w) (q : w = x) (r : x = y) (s : y = z)
  : whiskerL p (concat_p_pp q r s)
      @ concat_p_pp p (q@r) s
      @ whiskerR (concat_p_pp p q r) s
  = concat_p_pp p q (r@s) @ concat_p_pp (p@q) r s.
Proof.
  case p, q, r, s.  reflexivity.
Defined.
```

```
(** The 3-cell witnessing the left unit triangle. *)
Definition triangulator {A : Type} {x y z : A} (p : x = y) (q : y = z)
  : concat_p_pp p 1 q @ whiskerR (concat_p1 p) q
  = whiskerL p (concat_1p q).
Proof.
  case p, q.  reflexivity.
Defined.

(** The Eckmann-Hilton argument *)
Definition eckmann_hilton {A : Type} {x:A} (p q : 1 = 1 :> (x = x)) : p @ q = q @ p :=
  (whiskerR_p1 p @@ whiskerL_1p q)^
  @ (concat_p1 _ @@ concat_p1 _)
  @ (concat_1p _ @@ concat_1p _)
  @ (concat_whisker _ _ _ _ p q)
  @ (concat_1p _ @@ concat_1p _)^
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Formalization of mathematics

- The idea of logical foundations of math has great conceptual and philosophical interest, but in the past this was too lengthy and cumbersome to be of much use.

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- UF uses a "synthetic" method involving high-level axiomatics and structural descriptions. Allows for shorter, more abstract proofs that are closer to mathematical practice than the "analytic" method of ZFC.
- Software verification should also benefit from higher dimensional methods: current work in progress at CMU.

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- Other areas are also being developed:
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  - Elementary mathematics: basic algebra, real numbers, cardinal arithmetic, ...
- Some new logical ideas are suggested by the homotopy interpretation: Higher inductive types, Univalence axiom.

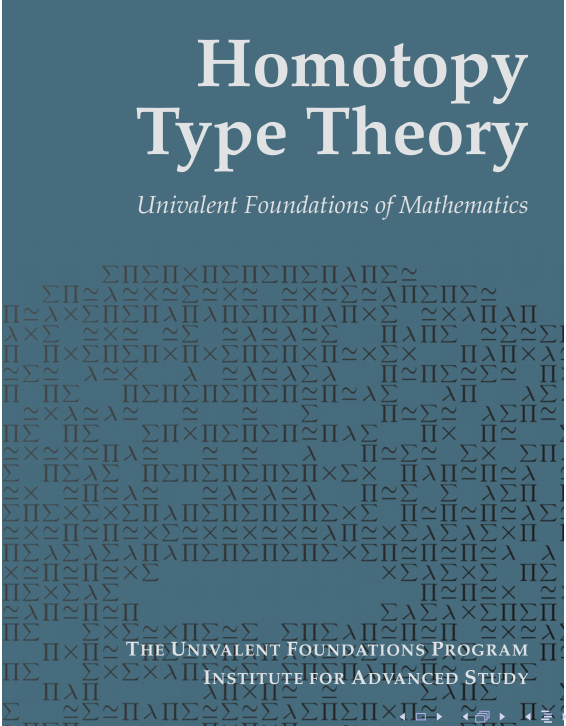
References and Further Information

More Information:

`www.HomotopyTypeTheory.org`

The Book:

Homotopy Type Theory:
Univalent Foundations of Mathematics


Homotopy
Type Theory
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Higher inductive types (Lumsdaine-Shulman)

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$$\begin{aligned} \mathrm{rec}(a, f)(0) &= a \\ \mathrm{rec}(a, f)(sn) &= f(\mathrm{rec}(a, f)(n)) \end{aligned}$$

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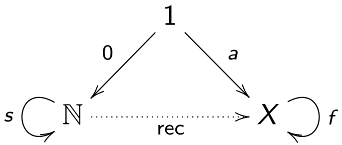
In other words, $(\mathbb{N}, 0, s)$ is the **free** structure of this type:



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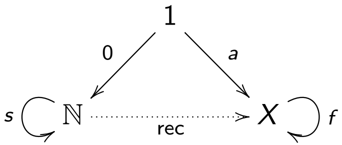

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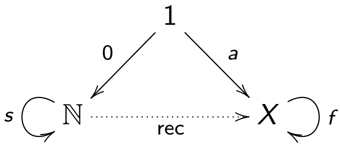

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Theorem

$\mathbb{N}$ *is a set (i.e. a* 0*-type).*

Higher inductive types: The circle S^1

The homotopical circle $\mathbb{S} = S^1$ can be given as an inductive type involving a "higher-dimensional" generator:

$$\mathbb{S} := \begin{cases} \mathsf{base} : \mathbb{S} \\ \mathsf{loop} : \mathtt{Id}_{\mathbb{S}}(\mathsf{base}, \mathsf{base}) \end{cases}$$

where we think of $\mathsf{loop} : \mathtt{Id}_{\mathbb{S}}(\mathsf{base}, \mathsf{base})$ as a path

$$\mathsf{loop} : \mathsf{base} \rightsquigarrow \mathsf{base}.$$

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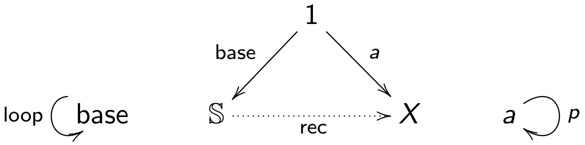

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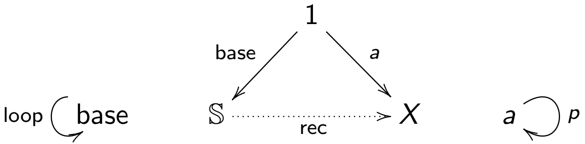

The map $\mathsf{rec}(a, p) : \mathbb{S} \to X$ is unique up to homotopy.

Higher inductive types: The circle S^1

Here is a sanity check:

Theorem (Shulman 2011)

The type-theoretic circle $\mathbb{S}$ has the correct homotopy groups:

$$\pi_n(\mathbb{S}) = \begin{cases} \mathbb{Z}, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases}$$

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The proof combines classical homotopy theory with methods from constructive type theory, and uses Voevodsky's Univalence Axiom. It has been formalized in Coq.

Higher inductive types: The interval I

The unit interval $\mathbb{I} = [0, 1]$ is also an inductive type, on the data:

$$\mathbb{I} := \begin{cases} 0, 1 : \mathbb{I} \\ p : \mathtt{Id}_{\mathbb{I}}(0, 1) \end{cases}$$

now thinking of $p : \mathtt{Id}_{\mathbb{I}}(0, 1)$ as a "free path"

$$p : 0 \rightsquigarrow 1.$$

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In topology, we start with the **interval** and use it to define the notion of a **path**.

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In HoTT, we start with the notion of a **path**, and use it to define the **interval**.

Higher inductive types: Conclusion

Many basic spaces and constructions can be introduced as HITs:

- higher spheres S^n, cylinders, tori, cell complexes, ...,
- suspensions ΣA,
- homotopy pullbacks, pushouts, etc.,
- truncations, such as connected components $\pi_0(A)$ and "bracket" types $[A]$,
- quotients by equivalence relations and general quotients,
- free algebras, algebras for a monad,
- (higher) homotopy groups π_n, Eilenberg-MacLane spaces $K(G, n)$, Postnikov systems,
- Quillen model structure,
- real numbers,
- cumulative hierarchy of sets.

Univalence

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- It is **formally incompatible** with the assumption that all types are **sets**.
- Its status as a **constructive principle** is the focus of much current research.

Isomorphism and Equivalence

In type theory, the usual notion of *isomorphism* $A \cong B$ is definable:

$$A \cong B \;\Leftrightarrow\; \textit{there are } f : A \to B \textit{ and } g : B \to A$$
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Formally, there is the type of isomorphisms:

$$\mathrm{Iso}(A, B) \;:=\; \sum_{f:A\to B} \sum_{g:B\to A} \Big(\prod_{x:A} \mathtt{Id}_A(gf(x), x) \times \prod_{y:B} \mathtt{Id}_B(fg(y), y) \Big)$$

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Thus $A \cong B$ iff this type is inhabited by a closed term, which is then just an isomorphism between A and B.

Isomorphism and Equivalence

- There is also a more refined notion of *equivalence* of types,

$$A \simeq B$$

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  which adds a further "coherence" condition relating the *proofs* of $gf(x) = x$ and $fg(y) = y$.
- Under the homotopy interpretation, this is the type of *homotopy equivalences*.
- This subsumes *categorical equivalence* (for 1-types), *isomorphism of sets* (for 0-types), and *logical equivalence* (for (-1)-types).

Invariance

One can show that all *definable properties* $P(X)$ of types X *respect type equivalence*:

$$A \simeq B \text{ and } P(A) \text{ implies } P(B)$$

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How is this related to **identity of types** A and B?

Univalence

To reason about **identity of types**, we need a *type universe* $\mathcal{U}$, with an identity type,

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So UA can be stated: *"Identity is equivalent to equivalence."*

The Univalence Axiom: Remarks

- Since UA is an equivalence, there is a map coming back:

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  In this sense, **equivalent objects are identical**.
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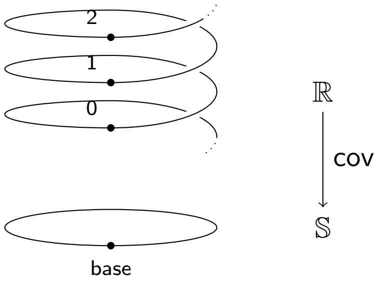
- So logically equivalent propositions are identical, and isomorphic sets, groups, etc., can be identified.
- UA implies that $\mathcal{U}$, in particular, is not a set (0-type).
- The computational character of UA is still an open question.

The Univalence Axiom: How it works

To compute the fundamental group of the circle $\mathbb{S}$, we shall construct the universal cover:



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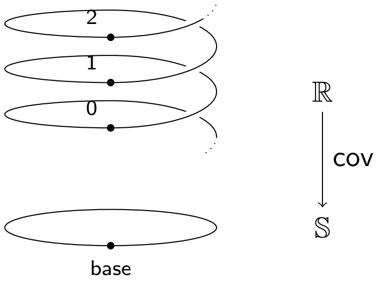

This will be a dependent type over $\mathbb{S}$, i.e. a type family

$$\mathsf{cov} : \mathbb{S} \longrightarrow \mathcal{U}.$$

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To define a type family

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by the recursion property of the circle, we just need the following data:

- a point $A : \mathcal{U}$
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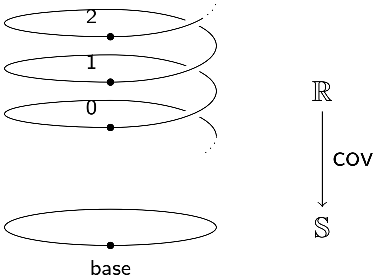
By Univalence, to give a loop $p : \mathbb{Z} \rightsquigarrow \mathbb{Z}$ in $\mathcal{U}$, it suffices to give an equivalence $\mathbb{Z} \simeq \mathbb{Z}$.
Since $\mathbb{Z}$ is a set, equivalences are just isomorphisms, so we can take the successor function $\mathsf{succ} : \mathbb{Z} \cong \mathbb{Z}$.

The Univalence Axiom: How it works



Definition (Universal Cover of $\mathbb{S}^1$)

The dependent type $\mathsf{cov} : \mathbb{S} \longrightarrow \mathcal{U}$ is given by circle-recursion, with

$$\mathsf{cov}(\mathsf{base}) := \mathbb{Z}$$
$$\mathsf{cov}(\mathsf{loop}) := \mathsf{ua}(\mathsf{succ}).$$

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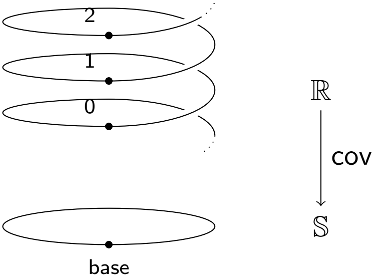

As in classical homotopy theory, we use the universal cover to define the "winding number" of any path $p : \mathsf{base} \rightsquigarrow \mathsf{base}$ by $\mathsf{wind}(p) = p_*(0)$.

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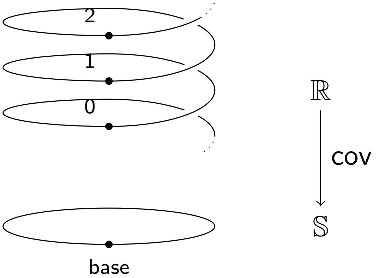

As in classical homotopy theory, we use the universal cover to define the "winding number" of any path $p : \mathsf{base} \rightsquigarrow \mathsf{base}$ by $\mathsf{wind}(p) = p_*(0)$. This gives a map

$$\mathsf{wind} : \Omega(\mathbb{S}) \longrightarrow \mathbb{Z},$$

which is inverse to the map $\mathbb{Z} \longrightarrow \Omega(\mathbb{S})$ given by

$$z \mapsto \mathsf{loop}^z.$$

The formal proof

```
(** * Theorems about the circle S^1. *)

Require Import Overture PathGroupoids Equivalences Trunc HSet.
Require Import Paths Forall Arrow Universe Empty Unit.
Local Open Scope path_scope.
Local Open Scope equiv_scope.
Generalizable Variables X A B f g n.

(* *** Definition of the circle. *)

Module Export Circle.

Local Inductive S1 : Type :=
| base : S1.

Axiom loop : base = base.

Definition S1_rect (P : S1 -> Type) (b : P base) (l : loop # b = b)
  : forall (x:S1), P x
  := fun x => match x with base => b end.

Axiom S1_rect_beta_loop
  : forall (P : S1 -> Type) (b : P base) (l : loop # b = b),
  apD (S1_rect P b l) loop = l.

End Circle.
```

```
(* *** The non-dependent eliminator *)

Definition S1_rectnd (P : Type) (b : P) (l : b = b)
  : S1 -> P
  := S1_rect (fun _ => P) b (transport_const _ _ @ l).

Definition S1_rectnd_beta_loop (P : Type) (b : P) (l : b = b)
  : ap (S1_rectnd P b l) loop = l.
Proof.
  unfold S1_rectnd.
  refine (cancelL (transport_const loop b) _ _ _).
  refine ((apD_const (S1_rect (fun _ => P) b _) loop)^ @ _).
  refine (S1_rect_beta_loop (fun _ => P) _ _).
Defined.

(* *** The loop space of the circle is the Integers. *)

(* First we define the appropriate integers. *)

Inductive Pos : Type :=
| one : Pos
| succ_pos : Pos -> Pos.

Definition one_neq_succ_pos (z : Pos) : ~ (one = succ_pos z)
  := fun p => transport (fun s => match s with one => Unit | succ_pos t => Empty end) p tt.

Definition succ_pos_injective {z w : Pos} (p : succ_pos z = succ_pos w) : z = w
  := transport (fun s => z = (match s with one => w | succ_pos a => a end)) p (idpath z).

Inductive Int : Type :=
| neg : Pos -> Int
| zero : Int
| pos : Pos -> Int.
```

```
Definition neg_injective {z w : Pos} (p : neg z = neg w) : z = w
  := transport (fun s => z = (match s with neg a => a | zero => w | pos a => w end)) p (idpath z).

Definition pos_injective {z w : Pos} (p : pos z = pos w) : z = w
  := transport (fun s => z = (match s with neg a => w | zero => w | pos a => a end)) p (idpath z).

Definition neg_neq_zero {z : Pos} : ~ (neg z = zero)
  := fun p => transport (fun s => match s with neg a => z = a | zero => Empty
  | pos _ => Empty end) p (idpath z).

Definition pos_neq_zero {z : Pos} : ~ (pos z = zero)
  := fun p => transport (fun s => match s with pos a => z = a
  | zero => Empty | neg _ => Empty end) p (idpath z).

Definition neg_neq_pos {z w : Pos} : ~ (neg z = pos w)
  := fun p => transport (fun s => match s with neg a => z = a
  | zero => Empty | pos _ => Empty end) p (idpath z).

(* And prove that they are a set. *)

Instance hset_int : IsHSet Int.
Proof.
  apply hset_decidable.
  intros [n | | n] [m | | m].
  revert m; induction n as [|n IHn]; intros m; induction m as [|m IHm].
  exact (inl 1).
  exact (inr (fun p => one_neq_succ_pos _ (neg_injective p))).
  exact (inr (fun p => one_neq_succ_pos _ (symmetry _ _ (neg_injective p)))).
  destruct (IHn m) as [p | np].
  exact (inl (ap neg (ap succ_pos (neg_injective p)))).
  exact (inr (fun p => np (ap neg (succ_pos_injective (neg_injective p))))).
  exact (inr neg_neq_zero).
  exact (inr neg_neq_pos).
  exact (inr (neg_neq_zero o symmetry _ _)).
  exact (inl 1).
```

```
  exact (inr (pos_neq_zero o symmetry _ _)).
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  exact (inl (ap pos (ap succ_pos (pos_injective p)))).
  exact (inr (fun p => np (ap pos (succ_pos_injective (pos_injective p))))).
Defined.

(* Successor is an autoequivalence of [Int]. *)

Definition succ_int (z : Int) : Int
  := match z with
       | neg (succ_pos n) => neg n
       | neg one => zero
       | zero => pos one
       | pos n => pos (succ_pos n)
     end.

Definition pred_int (z : Int) : Int
  := match z with
       | neg n => neg (succ_pos n)
       | zero => neg one
       | pos one => zero
       | pos (succ_pos n) => pos n
     end.

Instance isequiv_succ_int : IsEquiv succ_int
  := isequiv_adjointify succ_int pred_int _ _.
Proof.
  intros [[|n] | | [|n]]; reflexivity.
  intros [[|n] | | [|n]]; reflexivity.
Defined.
```

```
(* Now we do the encode/decode. *)

Section AssumeUnivalence.
Context `{Univalence} `{Funext}.

Definition S1_code : S1 -> Type
  := S1_rectnd Type Int (path_universe succ_int).

(* Transporting in the codes fibration is the successor autoequivalence. *)

Definition transport_S1_code_loop (z : Int)
  : transport S1_code loop z = succ_int z.
Proof.
  refine (transport_compose idmap S1_code loop z @ _).
  unfold S1_code; rewrite S1_rectnd_beta_loop.
  apply transport_path_universe.
Defined.

Definition transport_S1_code_loopV (z : Int)
  : transport S1_code loop^ z = pred_int z.
Proof.
  refine (transport_compose idmap S1_code loop^ z @ _).
  rewrite ap_V.
  unfold S1_code; rewrite S1_rectnd_beta_loop.
  rewrite <- path_universe_V.
  apply transport_path_universe.
Defined.
```

```
(* Encode by transporting *)

Definition S1_encode (x:S1) : (base = x) -> S1_code x
  := fun p => p # zero.

(* Decode by iterating loop. *)

Fixpoint loopexp {A : Type} {x : A} (p : x = x) (n : Pos) : (x = x)
  := match n with
       | one => p
       | succ_pos n => loopexp p n @ p
     end.

Definition looptothe (z : Int) : (base = base)
  := match z with
       | neg n => loopexp (loop^) n
       | zero => 1
       | pos n => loopexp (loop) n
     end.

Definition S1_decode (x:S1) : S1_code x -> (base = x).
Proof.
  revert x; refine (S1_rect (fun x => S1_code x -> base = x) looptothe _).
  apply path_forall; intros z; simpl in z.
  refine (transport_arrow _ _ _ @ _).
  refine (transport_paths_r loop _ @ _).
  rewrite transport_S1_code_loopV.
  destruct z as [[|n] | | [|n]]; simpl.
  by apply concat_pV_p.
  by apply concat_pV_p.
  by apply concat_Vp.
  by apply concat_1p.
  reflexivity.
Defined.
```

```
(* The nontrivial part of the proof that decode and encode are equivalences is showing that decoding
followed by encoding is the identity on the fibers over [base]. *)

Definition S1_encode_looptothe (z:Int)
  : S1_encode base (looptothe z) = z.
Proof.
  destruct z as [n | | n]; unfold S1_encode.
  induction n; simpl in *.
  refine (moveR_transport_V _ loop _ _ _).
  by apply symmetry, transport_S1_code_loop.
  rewrite transport_pp.
  refine (moveR_transport_V _ loop _ _ _).
  refine (_ @ (transport_S1_code_loop _)^).
  assumption.
  reflexivity.
  induction n; simpl in *.
  by apply transport_S1_code_loop.
  rewrite transport_pp.
  refine (moveR_transport_p _ loop _ _ _).
  refine (_ @ (transport_S1_code_loopV _)^).
  assumption.
Defined.
```

```
(* Now we put it together. *)

Definition S1_encode_isequiv (x:S1) : IsEquiv (S1_encode x).
Proof.
  refine (isequiv_adjointify (S1_encode x) (S1_decode x) _ _).
  (* Here we induct on [x:S1].  We just did the case when [x] is [base]. *)
  refine (S1_rect (fun x => Sect (S1_decode x) (S1_encode x))
    S1_encode_looptothe _ _).
  (* What remains is easy since [Int] is known to be a set. *)
  by apply path_forall; intros z; apply set_path2.
  (* The other side is trivial by path induction. *)
  intros []; reflexivity.
Defined.

Definition equiv_loopS1_int : (base = base) <~> Int
  := BuildEquiv _ _ (S1_encode base) (S1_encode_isequiv base).

End AssumeUnivalence.
```

Final Example: The cumulative hierarchy

Given a universe $\mathcal{U}$, we can make the *cumulative hierarchy* V of sets in $\mathcal{U}$ as a HIT:

- for any small A and any map $f : A \to V$, there is a "set":

  $$\mathsf{set}(A, f) : V$$

  We think of $\mathsf{set}(A, f)$ as the image of A under f, i.e. the classical set $\{f(a) \mid a \in A\}$
- For all $A, B : \mathcal{U}$, $f : A \to V$ and $g : B \to V$ such that

  $$\big(\forall a : A \, \exists b : B \;\, f(a) = g(b)\big) \wedge \big(\forall b : B \, \exists a : A \;\, f(a) = g(b)\big)$$

  we put in a path in V from $\mathsf{set}(A, f)$ to $\mathsf{set}(B, g)$.
- The 0-truncation constructor: for all $x, y : V$ and $p, q : x = y$, we have $p = q$.

The cumulative hierarchy of sets

Membership $x \in y$ is then defined for elements of V by:

$$(x \in \mathrm{set}(A, f)) \;:=\; (\exists a : A.\ x = f(a)).$$

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The proofs make essential use of UA.